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THE INITIAL BOUNDARY VALUE PROBLEM FOR THE EQUATIONS OF MOTION --ETC(U)

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THE INITIAL BOUNDARY VALUE PROBLEM
FOR THE EQUATIONS OF MOTION OF
COMPRESSIBLE VISCOUS AND
HEAT-CONDUCTIVE FLUID

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ABSTRACT

We prove the global existence and uniqueness of solutions to the equations of motion for compressible, viscous and heat-conductive Newtonian fluid in a bounded domain, with small initial data and external force, and boundary conditions of zero velocity and constant temperature. We also show that the solution decays exponentially to a unique equilibrium state. The proof uses an energy method similar to the one used in our previous results on the pure initial value problem plus some new techniques for estimates near the boundary.

AMS (MOS) Subject Classifications: 35B40, 35K60, 35M05, 76N10.

Key Words: Initial boundary value problem, Equations of motion, Compressible viscous and heat-conductive fluid, Global solutions in time, Energy method.

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

The motion of compressible, viscous and heat-conductive Newtonian fluid is described by a system of partial differential equations which is of hyperbolic-parabolic type and highly nonlinear. Having previously treated the pure initial value problem for these equations (see, e.g., MRC TSR #1991, 2194), we now consider the initial boundary value problem for motion within a smooth bounded container whose walls are kept at constant temperature, subject to a conservative external force, such as gravity. For this problem we prove the existence of a unique smooth solution for all positive time, and we show that this solution must decay exponentially to a unique equilibrium state. Since the system is quasilinear with respect to the unknowns, density, velocity and temperature, we need to assume that the initial data are close to the equilibrium state, and that the external force is sufficiently small.

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THE INITIAL BOUNDARY VALUE PROBLEM FOR THE EQUATIONS
OF MOTION OF COMPRESSIBLE VISCOUS AND
HEAT-CONDUCTIVE FLUID

Akitaka Matsumura* and Takaaki Nishida**

§1. Introduction. Statement of Theorem.

The motion of a compressible, viscous and heat-conductive Newtonian fluid is described by five conservation laws:

$$(1.1) \quad \left\{ \begin{array}{l} \rho_t + (\rho u^j)_{x_j} = 0, \\ u^i_t + u^j u^i_{x_j} + \frac{1}{\rho} p_{x_i} = \frac{1}{\rho} (\mu (u^i_{x_j} + u^j_{x_i}) + \mu' (u^k_{x_k}) \delta^{ij})_{x_j} \\ \quad + f^i, \quad i = 1, 2, 3, \\ \theta_t + u^j \theta_{x_j} + \frac{\theta p \theta}{\rho c_v} u^j_{x_j} = \frac{1}{\rho c_v} ((\kappa \theta_{x_j})_{x_j} + \Psi), \end{array} \right.$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geq 0$, ρ is the density, $u = (u^1, u^2, u^3)$ is the velocity, θ is the absolute temperature, $p = p(\rho, \theta)$ is the pressure, $f = (f^1, f^2, f^3)$ is the external force, $\mu = \mu(\rho, \theta)$ and $\mu' = \mu'(\rho, \theta)$ are viscosity coefficients, $\kappa = \kappa(\rho, \theta)$ is the coefficient of heat conductivity, $c_v = c_v(\rho, \theta)$ is the heat capacity at constant volume and $\Psi = \frac{1}{2} (u^j_{x_k} + u^k_{x_j})^2 + \mu' (u^j_{x_j})^2$ is the dissipation function. Let Ω be a bounded open set in \mathbb{R}^3 with a smooth boundary $\partial\Omega$. We consider the initial boundary value problem for (1.1) in Ω with the boundary conditions

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$$(1.2) \quad u(t, x) = 0, \quad \theta(t, x) = \bar{\theta}, \quad x \in \partial\Omega, \quad t \geq 0 ,$$

where $\bar{\theta}$ is any fixed positive constant; and with initial data

$$(1.3) \quad (\rho, u, \theta)(0, x) = (\rho_0, u_0, \theta_0)(x), \quad x \in \Omega .$$

We seek a classical solution for (1.1) ~ (1.3) globally in time.

Historically, a local solution in time for (1.1) ~ (1.3) was constructed by Tani [6]. On the other hand, Kazhikov and Shelukhin [3] proved the existence and uniqueness of a global solution in time for the one-space-dimensional model of (1.1) ~ (1.3). In this paper, we prove the existence and uniqueness of a global solution in time for the original problem (1.1) ~ (1.3) if the initial data and external force are suitably small. Although the proof is given by an energy method similar to those of our previous papers [4] and [5] on the initial value problem, the initial boundary value problem requires now a priori estimates of the solution near the boundary $\partial\Omega$. In order to state the main theorem precisely, let us list our assumptions:

(i) The external force f^i is generated by a potential function $\Phi(x)$,
i.e.,

$$f^i = -\Phi_{x_i} .$$

(ii) $\mu, \mu', \kappa, p, c_v \in C^{\infty}(0)$, where

$$O = \{(\rho, \theta) \mid (\rho, \theta) \in (0, +\infty) \times (0, +\infty)\}.$$

(iii) $\mu, \kappa, p, p_{\rho}, p_{\theta}, c_v > 0$ and $\mu' + \frac{2}{3}\mu > 0$ for $(\rho, \theta) \in O$.

(iv) $(\rho_0, u_0, \theta_0) \in H^4(\Omega)$, $\Phi \in H^5(\Omega)$ and $(\rho_0, \theta_0)(x) > 0$ for $x \in \bar{\Omega}$.

(v) (compatibility condition)

$$u_0|_{\partial\Omega} = 0, \quad \theta_0|_{\partial\Omega} = \bar{\theta} ,$$

$$-(p_0)_{x_i} + (\mu_0(u_0)_{x_j}^i + u_0)_{x_i}^j + u_0 u_0_{x_k}^k \delta^{ij})_{x_j} - p_0 \Phi_{x_j}|_{\partial\Omega} = 0 ,$$

$$-\theta_0(p_0)_{0} u_0_{x_j}^j + (\kappa_0 \theta_0)_{x_j} + \theta_0|_{\partial\Omega} = 0 ,$$

where $p_0 = p(p_0, \theta_0)$, $u_0 = u(p_0, \theta_0), \dots$, and so on. Here $H^k(\Omega)$ denotes the Sobolev's space on Ω with the norm $\| \cdot \|_k$, and $H_0^k(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in $H^k(\Omega)$. Define $\| \cdot \|_{C^k}$ by

$$\| u \|_{C^k} = \max_{|\alpha| \leq k} \sup_{x \in \Omega} \left| \left(\frac{\partial}{\partial x} \right)^\alpha u(x) \right| ,$$

and also define $\bar{\rho}$ by

$$\bar{\rho} = \frac{1}{V(\Omega)} \int_{\Omega} \rho_0(x) dx ,$$

where $V(\Omega)$ represents the volume of Ω . We say that $(\tilde{\rho}(x), \tilde{u}(x), \tilde{\theta}(x)) \in C^1(\bar{\Omega}) \times (C^2(\bar{\Omega}))^2$ is an equilibrium state of the problem (1.1) ~ (1.3) when $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ satisfies (1.1) and the following additional conditions

$$(1.4) \quad \tilde{u}|_{\partial\Omega} = 0 , \quad \tilde{\theta}|_{\partial\Omega} = \bar{\theta} , \quad \frac{1}{V(\Omega)} \int_{\Omega} \tilde{\rho} dx = \bar{\rho} .$$

Then we have the following main theorem.

Theorem 1.1. Under the assumptions (i) ~ (v), there exist positive constants ϵ, α and C such that if $\|\rho_0 - \bar{\rho}\|, \|u_0\|_4 + \|\theta_0 - \bar{\theta}\|_5 < \epsilon$, the problem (1.1) ~ (1.3) has a unique global solution in time (ρ, u, θ) and a unique equilibrium state $(\tilde{\rho}, \tilde{u}, \tilde{\theta}) = (\tilde{\rho}, 0, \bar{\theta})$ satisfying

$$\begin{aligned} \rho &\in C^0(0, +\infty; H^4(\Omega)) \cap C^1(0, +\infty; H^3(\Omega)) , \\ (u, \theta) &\in C^0(0, +\infty; H^4(\Omega)) \cap C^1(0, +\infty; H^2(\Omega)) , \end{aligned}$$

and

$$\|(\rho, u, \theta)(t) - (\tilde{\rho}, 0, \bar{\theta})\|_{C^0} \leq Ce^{-\alpha t} .$$

§2. Existence and Uniqueness of the Equilibrium State.

Let us rewrite the equations and conditions for the equilibrium state $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$:

$$(2.1) \quad (\tilde{\rho} \tilde{u}^j)_{x_j} = 0 ,$$

$$(2.2) \quad \tilde{\rho} u^j \tilde{u}_{x_j}^i + \tilde{p}_{x_i} + \tilde{\rho} \tilde{\theta}_{x_i} - (\tilde{\mu} (\tilde{u}_{x_j}^i + \tilde{u}_{x_i}^j) + \tilde{\mu} u_{x_k}^k \delta^{ij})_{x_j} = 0 ,$$

$$(2.3) \quad \tilde{\rho} \tilde{c}_v u^j \tilde{\theta}_{x_j} + \tilde{\theta} p_{\theta} u^j_{x_j} - (\tilde{\kappa} \tilde{\theta}_{x_j})_{x_j} - \tilde{\psi} = 0 ,$$

$$(2.4) \quad \tilde{u}|_{\partial\Omega} = 0 , \quad \tilde{\theta}|_{\partial\Omega} = \bar{\theta} , \quad \int \tilde{\rho} dx = \int \bar{\rho} dx ,$$

where $\tilde{p} = p(\tilde{\rho}, \tilde{\theta})$, $\tilde{\mu} = \mu(\tilde{\rho}, \tilde{\theta})$, ..., and so on. Then we have

Lemma 2.1. Given the assumptions (i) ~ (v), there exist positive constants ϵ and C such that if $\|\Phi\|_{C^1} < \epsilon$, the problem (2.1) ~ (2.4) has a unique solution $(\tilde{\rho}(x), 0, \bar{\theta})$ in a small neighborhood of $(\bar{\rho}, 0, \bar{\theta})$ in $C^1(\bar{\Omega}) \times (C^2(\bar{\Omega}))^2$ satisfying

$$\|\tilde{\rho} - \bar{\rho}\|_{C^1} < C \|\Phi\|_{C^1} ,$$

where $\tilde{\rho}(x)$ is determined by

$$(2.5) \quad \begin{cases} \text{Const.} - \Phi(x) = \int \frac{\tilde{\rho}(s) p_{\rho}(s, \bar{\theta})}{\bar{\rho}} ds , \\ \int \tilde{\rho}(x) - \bar{\rho} dx = 0 . \end{cases}$$

Furthermore we have $\tilde{\rho} \in H^5(\Omega)$ and

$$(2.6) \quad \|\tilde{\rho} - \bar{\rho}\|_5 < C \|\Phi\|_5 \text{ for } \|\Phi\|_5 < \epsilon .$$

Proof of Lemma 2.1. Suppose $|\rho - \bar{\rho}|, |u|, |\theta - \bar{\theta}| < \frac{1}{2} \min(\bar{\rho}, \bar{\theta})$. Then we may estimate the equalities

$$(2.7) \quad \begin{cases} \int [2.1] \times \int \frac{\tilde{\rho} p_{\rho}(s, \bar{\theta})}{\bar{\rho}} ds dx = 0 , \\ \int (\tilde{\rho} u^i) [2.2] dx = 0 , \\ \int \tilde{\rho} (\tilde{\theta} - \bar{\theta}) [2.3] dx = 0 , \end{cases}$$

where [2.1], [2.2]ⁱ and [2.3] denote the left hand side of (2.1), (2.2)ⁱ and (2.3) respectively. After integrating (2.7) by parts and using Poincare's inequalities

$$(2.8) \quad \|u\| \leq C\|Du\|, \quad \|\theta - \bar{\theta}\| \leq C\|D\theta\|,$$

one can show that

$$(2.9) \quad \|Du\|^2, \|D\theta\|^2 \leq C(\|D\rho\|_{C^0} + \|\theta - \bar{\theta}\|_{C^0})(\|Du\|^2 + \|D\theta\|^2).$$

Here and in what follows, we denote by $D_z^k f$ for $k > 0$ and $z = (z_1, \dots, z_m)$ the vector function

$$D_z^k f = \{\partial_z^k f / \partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \dots \partial z_m^{\alpha_m} \text{ for all } |\alpha| = k\}$$

and in particular write $D^k f$ when $z = x$. Therefore, if $\|D\rho\|_{C^0}$ and $\|\theta - \bar{\theta}\|_{C^0}$ are suitably small, it follows from (2.4) and (2.9) that

$$(2.10) \quad \tilde{u} \equiv 0, \quad \tilde{\theta} \equiv \bar{\theta}.$$

Substituting (2.10) into (2.2), we have

$$(2.11) \quad \left(\int_0^{\rho} \frac{p_p(s, \bar{\theta})}{s} ds + \Phi \right)_{x_i} = 0,$$

which implies (2.5) immediately.

One may use the implicit function theorem once to see that the constant in (2.5) depends on Φ as a smooth functional on $H^5(\Omega)$, and again to prove (2.6).

§3. Local and Global Existence.

First let us rewrite the problem (1.1) ~ (1.3) by the change of variables $(\rho, u, \theta) \rightarrow (\rho + \tilde{\rho}, u, \theta + \bar{\theta})$ as follows:

$$(3.1) \quad \left\{ \begin{array}{l} \rho_t + u^j \rho_{x_j} + (\rho + \tilde{\rho}) u^j_{x_j} + \rho_{x_j} u^j = 0, \\ u^i_t + u^j u^i_{x_j} - \frac{1}{\rho + \tilde{\rho}} (\mu (u^i_{x_j} + u^j_{x_i}) + \mu' \mu^k_{x_k} \delta^{ij})_{x_j} + \\ + \frac{p_\rho}{\rho + \tilde{\rho}} \rho_{x_i} + \frac{p_\theta}{\rho + \tilde{\rho}} \theta_{x_i} = \frac{\tilde{p}_\rho (\rho + \tilde{\rho}, \theta + \bar{\theta})}{(\rho + \tilde{\rho}) p_\rho (\rho, \bar{\theta})} - 1) \theta_{x_i}, \\ \theta_t + u^j \theta_{x_j} + \frac{(\theta + \bar{\theta}) p_\theta}{(\rho + \tilde{\rho}) c_v} u^j_{x_j} = - \frac{1}{(\rho + \tilde{\rho}) c_v} ((\kappa \theta_{x_j})_{x_j} + \Psi), \end{array} \right.$$

$$(3.2) \quad (u, \theta)|_{\partial\Omega} = 0,$$

$$(3.3) \quad (\rho, u, \theta)(x, 0) = (\rho_0, u_0, \theta_0)(x).$$

Furthermore we rewrite the problem (3.1) ~ (3.3) as follows:

$$(3.4)^0 \quad L_u^0(\rho, u) \equiv \rho_t + u^j \rho_{x_j} + \bar{\rho} u^j_{x_j} = f^0,$$

$$(3.4)^i \quad L^i(\rho, u, \theta) \equiv u^i_t - \mu u^i_{x_j x_j} - (\mu + \mu') u^j_{x_i x_j} + \\ + p_1 \rho_{x_i} + p_2 \theta_{x_i} = f^i,$$

$$(3.4)^4 \quad L^4(u, \theta) \equiv \theta_t - \kappa \theta_{x_j x_j} + p_3 u^j_{x_j} = f^4,$$

$$(3.5) \quad (u, \theta)|_{\partial\Omega} = 0, \quad (\rho, u, \theta)(0) = (\rho_0, u_0, \theta_0),$$

where define $L_v^0(\rho, u) \equiv \rho_t + v^j \rho_{x_j} + \bar{\rho} u^j_{x_j}$ and $\mu = \bar{\mu}/\bar{\rho}$, $\mu' = \bar{\mu}'/\bar{\rho}$,

$p_1 = \bar{p}_\rho/\bar{\rho}$, $p_2 = \bar{p}_\theta/\bar{\rho}$, $p_3 = \bar{\theta} \bar{p}_\rho/\bar{\rho} \bar{c}_v$, $\kappa = \bar{\kappa}/\bar{\rho} \bar{c}_v$, $\bar{\mu} = \mu(\bar{\rho}, \bar{\theta})$, $\bar{\mu}' = \mu'(\bar{\rho}, \bar{\theta})$, $\bar{p}_\rho = p_\rho(\bar{\rho}, \bar{\theta})$ and so on, and

$$f^0(\rho, u, \theta) \equiv (\bar{\rho} - \rho - \tilde{\rho}) u^j_{x_j} - \tilde{\rho} u^j_{x_j},$$

$$\begin{aligned}
f^1(\rho, u, \theta) &\equiv u^j u^i_{x_j} + \left(\frac{\mu}{\rho + \tilde{\rho}} - \frac{\bar{\mu}}{\rho} \right) u^i_{x_j x_j} + \\
&+ \left(\frac{\mu + \mu^*}{\rho + \tilde{\rho}} - \frac{\bar{\mu} + \bar{\mu}^*}{\rho} \right) u^j_{x_i x_j} + \frac{1}{\rho + \tilde{\rho}} (u^i_{x_j} (u^i_{x_j} + u^j_{x_i}) + \\
&+ (u^i_{x_j} u^k_{x_k} \delta^{ij}) + \left(\frac{\bar{p}_\rho}{\rho} - \frac{p_\rho}{\rho + \tilde{\rho}} \right) \rho_{x_i} + \\
&+ \left(\frac{\bar{p}_\theta}{\rho} - \frac{p_\theta}{\rho + \tilde{\rho}} \right) \theta_{x_i} + \left(\frac{\bar{p}_p (p + \tilde{p}, \theta + \bar{\theta})}{(p + \tilde{p}) p_\rho (\rho, \theta)} - 1 \right) \phi_{x_i} , \\
f^4(\rho, u, \theta) &\equiv -u^j \theta_{x_j} + \left(\frac{\kappa}{(\rho + \tilde{\rho}) c_v} - \frac{\bar{\kappa}}{\rho c_v} \right) \theta_{x_j x_j} + \\
&+ \left(\frac{\bar{\theta} \bar{p}_\theta}{\rho c_v} - \frac{(\theta + \bar{\theta}) p_\theta}{(\rho + \tilde{\rho}) c_v} \right) u^j_{x_j} + \frac{1}{(\rho + \tilde{\rho}) c_v} (\kappa_{x_j} \theta_{x_j} + \Psi) .
\end{aligned}$$

Next, by Sobolev's lemma there exists a positive constant E_0 such that

$$\|f\|_{C^0} \leq \frac{1}{2} \min(\bar{\rho}, \bar{\theta}) \quad \text{for} \quad \|f\|_2 \leq E_0.$$

Then the solution of (3.1) ~ (3.3) is sought in the set of functions

$X(0, +\infty; E)$ for some $E \leq E_0$, where for $0 \leq t_1 \leq t_2 \leq +\infty$,

$$X(t_1, t_2; E) = \{(\rho, u, \theta)\}$$

$$\rho \in C^0(t_1, t_2; \mathbb{H}^4) \cap L^2(t_1, t_2; \mathbb{H}^4) ,$$

$$\rho_t \in C^0(t_1, t_2; \mathbb{H}^3) \cap L^2(t_1, t_2; \mathbb{H}^3) ,$$

$$\rho_{tt} \in C^0(t_1, t_2; \mathbb{H}^1) \cap L^2(t_1, t_2; \mathbb{H}^2) ,$$

$$(u, \theta) \in C^0(t_1, t_2; \mathbb{H}^4 \cap \mathbb{H}_0^1) \cap L^2(t_1, t_2; \mathbb{H}^5) ,$$

$$(u, \theta)_t \in C^0(t_1, t_2; \mathbb{H}^2 \cap \mathbb{H}_0^1) \cap L^2(t_1, t_2; \mathbb{H}^3) ,$$

$$(u, \theta)_{tt} \in C^0(t_1, t_2; L^2) \cap L^2(t_1, t_2; H_0^1) ,$$

$$N(t_1, t_2) < \epsilon \} .$$

Here $N(t_1, t_2)$ is defined by

$$\begin{aligned} N^2(t_1, t_2) = & \sup_{t_1 \leq t \leq t_2} (\|\rho(t)\|_4^2 + \|\rho_t(t)\|_3^2 + \|\rho_{tt}(t)\|_1^2 + \\ & + \|u(t)\|_4^2 + \|u_t(t)\|_2^2 + \|u_{tt}(t)\|_1^2) + \\ & + \int_{t_1}^{t_2} (\|\rho(\tau)\|_4^2 + \|\rho_t(\tau)\|_3^2 + \|\rho_{tt}(\tau)\|_2^2 + \|u(\tau)\|_5^2 + \\ & + \|u_t(\tau)\|_3^2 + \|u_{tt}(\tau)\|_1^2) d\tau . \end{aligned}$$

We can get a global solution by a combination of a local existence result and some a priori estimates for the solution in X .

Proposition 3.1. (local existence) Suppose the problem (3.1) ~ (3.3) has a unique solution $(\rho, u, \theta) \in X(0, h; E_0)$ for some $h > 0$ and then consider the problem (3.1) ~ (3.3) for $t > h$. Then there exist positive constants τ, ϵ_0 and C_0 ($\epsilon_0 \sqrt{1+C_0^2} < E_0$) which are independent of h such that if $N(h, h), \|\theta\|_5 < \epsilon_0$, the problem has a unique solution

$$(\rho, u, \theta) \in X(h, h+\tau; C_0 N(h, h)) .$$

Proposition 3.2. (a priori estimate) Suppose the problems (3.1) ~ (3.3) has a solution $(\rho, u, \theta) \in X(0, h; E_0)$ for some $h > 0$. Then there exist positive constants ϵ_1 and C_1 ($\epsilon_1 < \epsilon_0, \epsilon_1 C_1 < E_0$) which are independent of h such that if $N(0, h), \|\theta\|_5 < \epsilon_1$, it holds that $N(0, h) < C_1 N(0, 0)$.

Given Propositions 3.1 and 3.2, the global existence uniqueness of solutions may be proved as follows. Choose the initial date (ρ_0, u_0, θ_0) and the potential function Φ so small that

$$N(0,0) \leq \min(e_0, \frac{e_1}{c_0}, \frac{e_1}{c_1 \sqrt{1+c_0^2}}) ,$$

and $\|\phi\|_5 < e_1$. Then Proposition 3.1 with $h = 0$ gives a local solution

$$(\rho, u, \theta) \in X(0, \tau; C_0 N(0,0)) .$$

Since $C_0 N(0,0) \leq e_1 \leq e_0$, Proposition 3.2 with $h = \tau$ implies

$$N(0, \tau) \leq C_1 N(0,0) ,$$

and Proposition 3.1 with $h = \tau$ implies

$$(\rho, u, \theta) \in X(\tau, 2\tau; C_0 N(\tau, \tau)) ,$$

$$\in X(0, 2\tau; \sqrt{1+c_0^2} N(0, \tau)) .$$

Hence, since $\sqrt{1+c_0^2} N(0, \tau) \leq C_1 \sqrt{1+c_0^2} N(0,0) \leq e_1$, Proposition 3.2 with

$h = 2\tau$ implies

$$N(0, 2\tau) \leq C_1 N(0,0) ,$$

and Proposition 3.1 with $h = 2\tau$ gives

$$(\rho, u, \theta) \in X(2\tau, 3\tau; C_0 N(2\tau, 2\tau)) ,$$

$$\in X(0, 3\tau; \sqrt{1+c_0^2} N(0, 2\tau)) .$$

Repetition of this process yields:

Proposition 3.3 (global existence)

There exist positive constants ϵ and C such that if $N(0,0)$,

$\|\phi\|_5 < \epsilon$, the problem (3.1) ~ (3.3) has a unique solution

$$(\rho, u, \theta) \in X(0, +\infty; C N(0,0)) .$$

§4. Proof of Local Existence and Uniqueness.

We note only how to prove local existence and uniqueness for $h = 0$, and omit the details. Suppose $(n, v, \zeta) \in X(0, \tau; E_0)$. Then we first need to solve the following equations:

$$(4.1)^0 \quad L_v^0(\rho, u) = f^0(\eta, v, \zeta) ,$$

$$(4.1)^1 \quad L^1(\rho, u, \theta) = f^1(\eta, v, \zeta) ,$$

$$(4.4)^4 \quad L^4(u, \theta) = f^4(\eta, v, \zeta) ,$$

$$(4.2) \quad (u, \theta)|_{\partial\Omega} = 0, \quad (\rho, u, \theta)(0) = (\rho_0, u_0, \theta_0) ,$$

where we note that $\{f^i\}_{i=0}^4$ satisfy the conditions

$$f^0 \in C^0(0, \tau; H^3) \cap L^2(0, \tau; H^4) ,$$

$$f_t^0 \in C^0(0, \tau; L^2) \cap L^2(0, \tau; H^2) ,$$

$$f_{tt}^0 \in L^2(0, \tau; L^2) ,$$

$$f^1 \in C^0(0, \tau; H^2) \cap L^2(0, \tau; H^3) ,$$

$$f_t^1 \in C^0(0, \tau; L^2) \cap L^2(0, \tau; H^1) ,$$

$$f_{tt}^1 \in C^0(0, \tau; H^{-1}) , \quad 1 \leq i \leq 4 ,$$

where H^{-1} represents the dual space of H_0^1 .

To solve (4.1), (4.2), we may regard the equations (4.1) as

$$(4.2)^0 \quad \rho_t + v^j \rho_{x_j} = -\bar{\rho} u_{x_j}^j + f^0 ,$$

$$(4.2)^1 \quad u_t^i - \hat{\mu} u_{x_j x_j}^i - (\hat{\mu} + \hat{\mu}^*) u_{x_i x_j}^j = f^i - p_2 \theta_{x_1} - p_1 \rho_{x_1} ,$$

$$(4.2)^4 \quad \theta_t - \hat{\kappa} \theta_{x_j x_j} = f^4 - p_3 u_{x_j}^j ,$$

and use standard arguments about first-order hyperbolic equations for $(4.2)^0$ and second-order parabolic systems for $\{(4.2)^i\}_{i=1}^4$. Once we get the solution of the linear problem (4.1)(4.2), we may construct an approximation sequence $\{(\rho, u, \theta)^{(n)}\}_{n=0}^{+\infty}$ for the nonlinear problem (3.4)(3.5) as follows:

$$(\rho, u, \theta)^{(0)} \equiv (\rho_0, u_0, \theta_0) ,$$

and for $n > 1$,

$$\left\{ \begin{array}{l} L^0_{u^{(n-1)}}(\rho^{(n)}, u^{(n)}) = f^0(\rho^{(n-1)}, u^{(n-1)}, \theta^{(n-1)}), \\ L^i(\rho^{(n)}, u^{(n)}, \theta^{(n)}) = f^i(\rho^{(n-1)}, u^{(n-1)}, \theta^{(n-1)}), \\ L^4(u^{(n)}, \theta^{(n)}) = f^4(\rho^{(n-1)}, u^{(n-1)}, \theta^{(n-1)}), \\ (u^{(n)}, \theta^{(n)})|_{\partial\Omega} = 0 , \quad (\rho^{(n)}, u^{(n)}, \theta^{(n)})(0) = (\rho_0, u_0, \theta_0) . \end{array} \right.$$

Finally, we may apply the contraction mapping principle to the nonlinear problem (3.4), (3.5) in the space X under the smallness conditions on the initial data and external force, to obtain existence and uniqueness of the solution.

§5. Proof of the A Priori Estimates.

Throughout this section, we suppose that $\|\theta\|_5 \leq \epsilon_0$ and that the problem (3.1) ~ (3.3) has a solution $(\rho, u, \theta) \in X(0, h; E_0)$ and we note that all constants are independent of h . First, recall Poincare's inequalities for (u, θ) :

$$(5.1) \quad \|u\|^2 \leq C \|Du\|^2, \quad \|\theta\|^2 \leq C \|D\theta\|^2,$$

and the following lemma for ρ .

Lemma 5.1. It holds that

$$\|\rho\|^2 \leq C \|D\rho\|^2.$$

This is proved using the fact that the equation $(3.1)^0$ implies

$$\int \rho(t) dx = \int \rho_0 dx = 0$$

for all $t > 0$ (cf. [7]).

Next, since the viscosity and heat conduction terms of (3.4) define strongly elliptic operators on (u, θ) , we have for $(u, \theta) \in H^k \cap H_0^1$ ($1 \leq k \leq 5$) (cf. [1], [2]):

$$(5.2) \quad \begin{aligned} \|u\|_k^2 &\leq C \sum_{i=1}^3 \|\hat{u} u_{x_i x_j}^i + (\hat{u} + \hat{u}^*) u_{x_i x_j}^j\|_{k-2}^2, \\ \|\theta\|_k^2 &\leq C \|\hat{u} \theta_{x_j x_j}\|_{k-2}^2. \end{aligned}$$

Equations (3.4) and estimates (5.1) and (5.2) easily yield the following lemma.

Lemma 5.2. For $1 \leq k \leq 5$, it holds that

$$\left\{ \begin{aligned} \|u\|_k^2 &\leq C(\|u_t\|_{k-2}^2 + \|\theta\|_{k-1}^2 + \|\rho\|_{k-1}^2 + \sum_{i=1}^3 \|f^i\|_{k-2}^2), \\ \|\theta\|_k^2 &\leq C(\|\theta_t\|_{k-2}^2 + \|u\|_{k-1}^2 + \|f^4\|_{k-2}^2). \end{aligned} \right.$$

Third, the following estimates on the time derivatives of the solution over all of Ω can be obtained rather easily.

Lemma 5.3. For any positive number ϵ and $0 < k < 2$, it holds that

$$\begin{aligned} \|D_t^k(\rho, u, \theta)(t)\|^2 + \int_0^t \|D_t^k(u, \theta)\|_1^2 + \|D_t^k(\frac{d\rho}{dt})\|^2 d\tau \\ \leq C \|D_t^k(\rho, u, \theta)(0)\|^2 + \int_0^t \epsilon \|D_t^k \rho\|^2 + C(1 + \epsilon^{-1})(\|D_t^k f^0\|^2 + \\ + \sum_{i=1}^4 \|D_t^k f^i\|_{-1}^2) d\tau + C N^3(0, t) , \end{aligned}$$

where $\frac{d\rho}{dt}$ is defined by

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u^j \frac{\partial \rho}{\partial x_j} .$$

This lemma is proved by estimating the following equation derived from (3.4):

$$\begin{aligned} \int_0^t \int \frac{p_1}{\rho} D_t^k L^0 \cdot D_t^k \rho + D_t^k L^1 \cdot D_t^k u^i + \frac{p_2}{p_3} D_t^k L^4 \cdot D_t^k \theta dx d\tau \\ = \int_0^t \frac{p_1}{\rho} D_t^k f^0 \cdot D_t^k \rho + D_t^k f^1 \cdot D_t^k u^i + \frac{p_2}{p_3} D_t^k f^4 \cdot D_t^k \theta dx d\tau , \end{aligned}$$

and noting from (3.4)⁰ that

$$u^j_{x_j} = -\frac{1}{\rho} \frac{d\rho}{dt} + \frac{1}{\rho} f^0 .$$

Fourth, let us establish the interior estimates. Let x_0 be any fixed function in $\tilde{C}_0^\infty(\Omega)$. Then we have

Lemma 5.4. For $0 < \ell < 3$, it holds that

$$\begin{aligned} \|x_0 D_t^{\ell+1} \rho(t)\|^2 + \int_0^t \|x_0 D_t^{\ell+1} \rho\|^2 d\tau \leq \\ \leq C(\|x_0\|_{\ell+1}^2 + \|u_0\|_\ell^2 + \|u(t)\|_\ell^2 + \int_0^t \|u, \theta\|_{\ell+1}^2 + \end{aligned}$$

$$+ \| \varepsilon^0 \|_{k+1}^2 + \sum_{i=1}^3 \| \varepsilon^i \|_k^2 d\tau + N^3(0, t) \quad .$$

Lemma 5.5. For any positive number ϵ and $1 \leq k \leq 4$, it holds that

$$\begin{aligned} & \| x_0 D^k(\rho, u, \theta)(t) \|^2 + \int_0^t \| x_0 D^{k+1} u \|^2 + \| x_0 D^k \left(\frac{d\rho}{dt} \right) \|^2 d\tau \\ & \leq C \| \rho_0, u_0, \theta_0 \|_k^2 + \int_0^t \epsilon \| x_0 D^k \rho \|^2 + C(1 + \epsilon^{-1}) \\ & \quad \| u, \theta \|_k^2 + \| \varepsilon^0 \|_k^2 + \sum_{i=1}^4 \| \varepsilon^i \|_{k-1}^2 d\tau + CN^3(0, t) \quad . \end{aligned}$$

To prove these, we may estimate for Lemma 5.4 the equation

$$\begin{aligned} & \int_0^t \int x_0^2 (D^k L^0)_{x_i} \cdot D^k \rho_{x_i} + \frac{\bar{\rho}}{2\hat{\mu} + \hat{\mu}'} x_0^2 D^k L^1 \cdot D^k \rho_{x_i} dx d\tau \\ & - \int_0^t \int x_0^2 D^k f^0_{x_i} \cdot D^k \rho_{x_i} + \frac{\bar{\rho}}{2\hat{\mu} + \hat{\mu}'} x_0^2 D^k f^1 \cdot D^k \rho_{x_i} dx d\tau \quad , \end{aligned}$$

and for Lemma 5.5, the equation

$$\begin{aligned} & \int_0^t \int \frac{p_1}{\bar{\rho}} x_0^2 D^k L^0 \cdot D^k \rho + x_0^2 D^k L^1 \cdot D^k u^1 + \frac{p_2 x_0^2}{p_3} D^k L^4 \cdot D^k \theta dx d\tau \\ & - \int_0^t \int \frac{p_1}{\bar{\rho}} x_0^2 D^k f^0 \cdot D^k \rho + x_0^2 D^k f^1 \cdot D^k u^1 + \frac{p_2 x_0^2}{p_3} D^k f^4 \cdot D^k \theta dx d\tau \quad . \end{aligned}$$

Fifth, let us establish the estimates near the boundary. We choose a finite number of bounded open sets $\{O_i\}_{i=1}^N$ in \mathbb{R}^3 such that

$$\bigcup_{j=1}^N O_j = \partial\Omega \quad ,$$

and in each open set O_j we choose local coordinates (ψ, φ, r) as follows:

(i) the surface $O_j \cap \partial\Omega$ is the image of smooth functions

$y^i = y^i(\psi, \varphi)$ satisfying

$$(5.3) \quad \begin{cases} y_\psi^i \cdot y_\psi^i = 1, & y_\psi^i \cdot y_\varphi^i = 0, \\ y_\varphi^i \cdot y_\varphi^i > \delta > 0, \end{cases}$$

where δ is some positive constant independent of $1 \leq j \leq N$,

(ii) any x^i in O_j are represented by

$$(5.4) \quad x^i = x^i(\psi, \varphi, r) = r n^i(\psi, \varphi) + y^i(\psi, \varphi),$$

where $n^i(\psi, \varphi)$ represents the internal unit normal vector at the point of the surface coordinated (ψ, φ) .

Here and in what follows, we omit the suffix j for simplicity. Let us define the unit vectors e_1^i and e_2^i by

$$e_1^i = y_\psi^i, \quad e_2^i = y_\varphi^i / |y^i|.$$

Then note that there exist smooth functions of (ψ, φ) , $(\alpha, \beta, \gamma)(\psi, \varphi)$ and $(\alpha^i, \beta^i, \gamma^i)(\psi, \varphi)$ satisfying

$$\frac{\partial}{\partial \psi} \begin{pmatrix} e_1^i \\ e_2^i \\ n^i \end{pmatrix} = \begin{pmatrix} 0 & -\gamma & -\alpha \\ \gamma & 0 & -\beta \\ \alpha & \beta & 0 \end{pmatrix} \begin{pmatrix} e_1^i \\ e_2^i \\ n^i \end{pmatrix},$$

$$\frac{\partial}{\partial \varphi} \begin{pmatrix} e_1^i \\ e_2^i \\ e_3^i \end{pmatrix} = \begin{pmatrix} 0 & -\gamma^i & -\alpha^i \\ \gamma^i & 0 & -\beta^i \\ \alpha^i & \beta^i & 0 \end{pmatrix} \begin{pmatrix} e_1^i \\ e_2^i \\ e_3^i \end{pmatrix}.$$

An elementary calculation will show that the Jacobian J of the transformation (5.4) is

$$(5.6) \quad \begin{aligned} J &= |x_\psi \times x_\varphi| \\ &= |y_\varphi| + (\alpha|y_\varphi| + \beta^*)r + (\alpha\beta^* - \beta\alpha^*)r^2. \end{aligned}$$

By (5.6), we can see the transformation (5.4) is regular choosing r small if needed. Therefore the functions $(\psi, \varphi, r)_{x_i}(x)$ make sense and are calculated as

$$(5.7) \quad \left\{ \begin{aligned} \psi_{x_i} &= \frac{1}{J} (x_\psi \times x_r)_i = \frac{1}{J} (Ae_1^i + Be_2^i), \\ \varphi_{x_i} &= \frac{1}{J} (x_r \times x_\varphi)_i = \frac{1}{J} (Ce_1^i + De_2^i), \\ r_{x_i} &= \frac{1}{J} (x_\psi \times x_\varphi)_i = n_i, \end{aligned} \right.$$

where $A = |y_\varphi| + \beta^*r$, $B = -r\alpha^*$, $C = -\beta r$, $D = 1 + \alpha r$ and $J = AD - BC > 0$.

Hence (5.7) implies

$$\frac{\partial}{\partial x_i} = \frac{1}{J} (Ae_1^i + Be_2^i) \frac{\partial}{\partial \psi} + \frac{1}{J} (Ce_1^i + De_2^i) \frac{\partial}{\partial \varphi} + n^i \frac{\partial}{\partial r}.$$

Thus, in each O_j , we can rewrite the equations $\{(4.1)_i^j\}_{i=0}^3$ in the local coordinates (ψ, φ, r) as follows:

$$\begin{aligned} \rho_t + \frac{1}{J} ((Ae_1^i n^i + Be_2^i u^i) \rho_\psi + (Ce_1^i + De_2^i) u^i \rho_\varphi + u^i n^i J \rho_r) \\ + \frac{\bar{\rho}}{J} ((Ae_1^i + Be_2^i) u_\psi^i + (Ce_1^i + De_2^i) u_\varphi^i + J n^i u_r^i) = f^0, \\ u_t^i + \frac{\hat{\mu}}{J^2} ((A^2 + B^2) u_{\psi\psi}^i + 2(AC + BD) u_{\psi\varphi}^i + (C^2 + D^2) u_{\varphi\varphi}^i + \end{aligned}$$

$$+ J^2 u_{rr}^1) + \text{first order terms of } u +$$

$$+ \frac{1}{J} (Ae_1^1 + Be_2^1) \left(\frac{\hat{\mu} + \hat{\mu}'}{\hat{\rho}} \frac{dp}{dt} + p_1 \rho \right) \psi +$$

$$+ \frac{1}{J} (Ce_1^1 + De_2^1) \left(\frac{\hat{\mu} + \hat{\mu}'}{\hat{\rho}} \frac{dp}{dt} + p_1 \rho \right) \varphi +$$

$$+ n^1 \left(\frac{\hat{\mu} + \hat{\mu}'}{\hat{\rho}} \frac{dp}{dt} + p_1 \rho \right)_r$$

$$= f^1 - p_2 \theta_{x_1} + \frac{\hat{\mu} + \hat{\mu}'}{\hat{\rho}} f_{x_1}^0 ,$$

where we note that $J^2 = (AC + BD)^2 - (A^2 + B^2)(C^2 + D^2)$. Now let χ be any fixed function in $C_0^\infty(0, J)$. By methods similar to those of Lemma 5.5, we can get

Lemma 5.6. For any positive number ϵ and $1 \leq k \leq 4$, it holds that

$$\begin{aligned} & \|\chi D_{\psi, \varphi}^k (\rho, u)(t)\|^2 + \int_0^t \|\chi D_{\psi, \varphi}^k D u\|^2 + \|\chi D_{\psi, \varphi}^k (\frac{dp}{dt})\|^2 dt \\ & \leq C \|\rho_0, u_0\|_k^2 + \int_0^t \epsilon \|\chi D_{\psi, \varphi}^k \rho\|^2 + C(1 + \epsilon^{-1}) (\\ & \quad \|u, \theta\|_k^2 + \|f^0\|_k^2 + \sum_{i=1}^3 \|f^i\|_{k-1}^2) d\tau + CN^3(0, t) . \end{aligned}$$

Moreover, by (5.8), we get:

$$\begin{aligned} (5.9) \quad & \frac{2\hat{\mu} + \hat{\mu}'}{\hat{\rho}} \left(\frac{dp}{dt} \right)_r + p_1 \rho_r = -n^1 u_t^1 + \frac{\hat{\mu}}{J^2} (\\ & (A^2 + B^2) n^1 u_{\psi\psi}^1 + 2(AC + BD) n^1 u_{\psi\varphi}^1 + (C^2 + D^2) n^1 u_{\varphi\varphi}^1 - \\ & - J(Ae_1^1 + Be_2^1) u_{\psi r}^1 - J(Ce_1^1 + De_2^1) u_{r\varphi}^1) + \end{aligned}$$

$$+ \text{first order terms of } (u, \theta) + n^1 f^1 + \frac{2\hat{\mu} + \hat{\mu}^0}{\hat{\rho}} f^0_r .$$

Estimating the equation

$$\int_0^t \int x^2 |D_{\psi, \varphi}^k D_x^l (\text{equation (5.9)})|^2 dx dt ,$$

for $0 \leq k+l \leq 3$, we have

Lemma 5.7. For $0 \leq k+l \leq 3$, it holds that

$$\begin{aligned} & \|x D_{\psi, \varphi}^k D_x^l \rho(t)\|^2 + \int_0^t \|x D_{\psi, \varphi}^k D_x^{l+1} \rho\|^2 + \\ & + \|x D_{\psi, \varphi}^k D_x^{l+1} (\frac{d\rho}{dt})\|^2 dt \leq C \| \rho_0 \|_{k+l+1}^2 + \\ & + C \int_0^t \|x D_{\psi, \varphi}^{k+1} D_x^l D u\|^2 + \|u_t\|_{k+l}^2 + \|u, \theta\|_{k+l+1}^2 + \\ & + \|f^0\|_{k+l+1}^2 + \sum_{i=1}^3 \|f^i\|_{k+l}^2 dt + C N^3(0, t) . \end{aligned}$$

Sixth, we note the following lemma about the linear stationary Stokes Equation (see [7], for example).

Lemma 5.8. Consider the problem

$$(5.10) \quad \begin{cases} \bar{\rho} u_{x_j}^j = g^0 , \\ -\mu u_{x_j x_j}^1 + p_1 \rho_{x_1} = g^1 , \\ u|_{\partial\Omega} = 0 , \end{cases}$$

where $g^0 \in H^{k+1}$ and $g^1 \in H^k$ ($k \geq 0$).

Then the problem (5.10) has a solution (ρ, u) such that $\rho \in H^{k+1}$ and $u \in H^{k+2} \cap H_0^1$ which is unique modulo a constant of integration for ρ ; this solution satisfies

$$\|u\|_{k+2}^2 + \|D\psi\|_k^2 \leq C(\|g^0\|_{k+1}^2 + \sum_{i=1}^3 \|g^i\|_k^2) .$$

Apply Lemma 5.8 to the equations $\{(3.4)\}_{i=0}^3$, we have

Lemma 5.9. For $2 \leq k \leq 5$, it holds that

$$\|u\|_k^2 + \|D\psi\|_{k-2}^2 \leq C(\|\frac{dp}{dt}\|_{k-1}^2 + \|u_t\|_{k-2}^2 +$$

$$+ \|\theta\|_{k-1}^2 + \sum_{i=1}^3 \|f^i\|_{k-2}^2 + \|f^0\|_{k-1}^2) .$$

Furthermore by differentiating the equations $\{(3.4)\}_{i=0}^3$ k times with respect to ψ and φ in each O_j , multiplying them by x and again applying Lemma 5.8, we obtain

Lemma 5.10. For $1 \leq l+k \leq 3$, it holds that

$$\begin{aligned} & \|x D^{2+l} D_{\psi, \varphi}^k u\|^2 + \|x D^{1+l} D_{\psi, \varphi}^k \rho\|^2 \\ & \leq C(\|x D^{1+l} D_{\psi, \varphi}^k (\frac{dp}{dt})\|^2 + \|u_t \theta\|_{l+k+1}^2 + \|\rho\|_{l+k}^2 + \\ & \quad + \|\rho_t\|_{l+k}^2 + \|f^0\|_{k+l+1}^2 + \sum_{i=1}^3 \|f^i\|_{k+l}^2) . \end{aligned}$$

Using all of the above preparations, we are ready to establish the desired a priori estimates. Although we omit the details, by combining Lemma 5.3 with $k = 0$ and 1, Lemma 5.2 with $k = 1, 2$ and 3, Lemma 5.4 with $l = 0$ and 1, Lemma 5.5 with $k = 1$ and 2, Lemma 5.6 with $k = 1$ and 2, Lemma 5.7 with $k = l = 0$ and $k+l = 1$, Lemma 5.9 with $k = 2$ and 3 and Lemma 5.10 with $l = 0$ and $k = 1$, and by choosing ϵ suitably small, we have

$$(5.11) \quad \|\rho, u, \theta\|_2^2 + \|u_t, \theta_t\|_1^2 + \|\rho_t\|_1^2 +$$

$$\begin{aligned}
& + \int_0^t \|u_t, \theta\|_3^2 + \|\rho\|_2^2 + \|\frac{d\rho}{dt}\|_2^2 + \|u_{tt}, \theta_t\|_1^2 + \|\frac{d\rho}{dt}\|_t^2 d\tau \\
& \leq C(\|\rho_0, u_0, \theta_0\|_2^2 + \sup_{0 \leq \tau \leq t} (\|f^0\|_1^2 + \sum_{i=1}^4 \|f^i\|_1^2) + \\
& + \int_0^t \|f^0\|_2^2 + \sum_{i=1}^4 \|f^i\|_1^2 + \|f_t^0\|_2^2 + \sum_{i=1}^4 \|f_t^i\|_{-1}^2 d\tau) \\
& + CN^3(0, t) .
\end{aligned}$$

In the same way, after considering the situation differentiated with respect to t , we also have

$$\begin{aligned}
(5.12) \quad & \|\rho, u, \theta\|_t^2 + \|u_{tt}, \theta_{tt}\|_1^2 + \|\rho_{tt}\|_1^2 + \\
& + \int_0^t \|u_t, \theta_t\|_3^2 + \|\frac{d\rho}{dt}\|_2^2 + \|\frac{d\rho}{dt}\|_t^2 + \|u_{tt}, \theta_{tt}\|_1^2 + \|\frac{d\rho}{dt}\|_{tt}^2 d\tau \\
& \leq C(\|\rho_0, u_0, \theta_0\|_4^2 + \sum_{i=0}^4 \|f^i(0)\|_2^2 + \sup_{0 \leq \tau \leq t} (\sum_{i=1}^4 \|f_t^i\|_1^2 + \|f_t^0\|_1^2) + \\
& + \int_0^t \|f_t^0\|_2^2 + \sum_{i=1}^4 \|f_t^i\|_1^2 + \|f_{tt}^0\|_2^2 + \sum_{i=1}^4 \|f_{tt}^i\|_{-1}^2 d\tau) + \\
& + CN^3(0, t) .
\end{aligned}$$

By beginning with the estimates (5.11) and (5.12) and combining the Lemmas 5.2 ~ 5.10 again, we can reach

$$\begin{aligned}
(5.13) \quad & \sup_{0 \leq \tau \leq t} (\|\rho, u, \theta\|_4^2 + \|u_t, \theta_t\|_2^2 + \|u_{tt}, \theta_{tt}\|_1^2 + \|\rho_t\|_3^2 + \\
& + \|\rho_{tt}\|_1^2) + \int_0^t \|u, \theta\|_5^2 + \|\rho\|_4^2 + \|u_t, \theta_t\|_3^2 + \|\frac{d\rho}{dt}\|_4^2 + \\
& + \|u_{tt}, \theta_{tt}\|_1^2 + \|\frac{d\rho}{dt}\|_t^2 + \|\frac{d\rho}{dt}\|_{tt}^2 d\tau \leq
\end{aligned}$$

$$\begin{aligned}
& \leq C(\|p_0, u_0, \theta_0\|_4^2 + \sup_{0 \leq \tau \leq t} (\|f^0\|_3^2 + \|f_t^0\|_1^2 + \sum_{i=1}^4 \|f_t^i\|_2^2 \\
& + \|f_t^1\|_2^2) + \int_0^t \|f^0\|_4^2 + \|f_t^0\|_2^2 + \|f_{tt}^0\|_2^2 + \sum_{i=1}^4 (\|f_t^i\|_3^2 + \|f_t^i\|_1^2 + \|f_{tt}^i\|_1^2) d\tau + N^3(0, t)) .
\end{aligned}$$

Therefore, noting that

$$\frac{dp}{dt} = p_t + \bar{p} u^j p_{x_j},$$

we have consequently

Lemma 5.11. It holds that

$$\begin{aligned}
(5.14) \quad N^2(0, t) & \leq C(\|p_0, u_0, \theta_0\|_4^2 + \sup_{0 \leq \tau \leq t} (\|f^0\|_3^2 + \|f_t^0\|_1^2 + \\
& + \sum_{i=1}^4 (\|f_t^i\|_2^2 + \|f_t^1\|_2^2)) + \int_0^t (\|f^0\|_4^2 + \|f_t^0\|_2^2 + \\
& \|f_{tt}^0\|_2^2 + \sum_{i=1}^4 (\|f_t^i\|_3^2 + \|f_t^i\|_1^2 + \|f_{tt}^i\|_1^2) d\tau + N^3(0, t)) .
\end{aligned}$$

Now let us establish the estimates for $\{f^i\}$.

Lemma 5.12. It holds that the right hand side of

$$(5.14) \leq C(\|p_0, u_0, \theta_0\|_4^2 + (N(0, t) + \|\theta\|_5)N^2(0, t)).$$

Here we note that Lemmas 5.11 and 5.12 easily imply the desired a priori estimates, in fact, we may choose ϵ_1 so small that

$$N^2(0, t) \leq C(\|p_0, u_0, \theta_0\|_4^2 \text{ for } N(0, t), \|\theta\|_5 \leq \epsilon_1).$$

Proof of Lemma 5.12. Because there are too many terms to estimate, we only give proofs for a few examples. The remaining terms can be proved in the same way.

Consider f^0 , for example, as follows:

$$\begin{aligned}
& \sup_{\tau} \|(\bar{\rho} - \rho - \tilde{\rho}) u_{x_j}^j - \tilde{\rho}_{x_j} u_j^j\|_3^2 \\
& \leq \sup_{\tau} (\|\rho\|_3^2 \|u\|_4^2 + \|\tilde{\rho} - \bar{\rho}\|_3^2 \|u\|_4^2 + \|D\tilde{\rho}\|_3^2 \|u\|_3^2) \\
& \leq C E_0(N(0, t) + \|\Phi\|_5) N^2(0, t) ,
\end{aligned}$$

$$\begin{aligned}
& \int_0^t \|(\bar{\rho} - \rho - \tilde{\rho}) u_{x_j}^j - \tilde{\rho}_{x_j} u_j^j\|_4^2 d\tau \\
& \leq \int_0^t \|\rho\|_4^2 \|u\|_5^2 + \|\tilde{\rho} - \bar{\rho}\|_4^2 \|u\|_5^2 + \|D\tilde{\rho}\|_4^2 \|u\|_4^2 d\tau \\
& \leq C E_0(N(0, t) + \|\Phi\|_5) N^2(0, t) , \dots
\end{aligned}$$

and so on. Proceeding in this manner proves Lemma 5.12. Finally, in order to prove Theorem 1.1, it suffices to show the exponential decay of the solution. To do that, we may estimate the equations

$$\left\{
\begin{aligned}
L_u^0(e^{\alpha t}(\rho, u)) &= e^{\alpha t} f^0 + \alpha e^{\alpha t} \rho , \\
L^i(e^{\alpha t}(\rho, u, \theta)) &= e^{\alpha t} f^i + \alpha e^{\alpha t} u^i , \\
L^4(e^{\alpha t}(u, \theta)) &= e^{\alpha t} f^4 + \alpha e^{\alpha t} \theta ,
\end{aligned}
\right.$$

in the same way as (5.11) and taking α suitably small.

REFERENCES

- [1] S. Agmon, A. Douglis, Nirenberg: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary condition I, Comm. Pure Appl. Math., 12, 1959, 623-727.
- [2] S. Agmon, A. Douglis, Nirenberg: Estimates near the boundary for solutions of elliptic partial differential equations satisfying boundary condition II, Comm. Appl. Math., 17, 1964, 35-92.
- [3] A. V. Vrazhikhov, V. V. Shelukkin: Unique global solution in time of initial-boundary value problems for one-dimensional equations of a viscous gas, Prikl. Mat. Mech., 41, 1977, 282-291.
- [4] A. Matsumura, T. Nishida: The initial value problem for the equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ., 20, 1980, 67-104.
- [5] A. Matsumura: An energy method for the equations of motion of compressible viscous and heat-conductive fluids, University of Wisconsin-Madison, MRC Technical Summary Report #2194.
- [6] A. Tani: On the first initial-boundary value problem of compressible viscous fluid motion, Publ. RIMS, Kyoto Univ., 13, 1977, 193-253.
- [7] R. Teman: Navier Stokes Equations, North-Holland Publishing Company-Amsterdam, New York, Oxford.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) We prove the global existence and uniqueness of solutions to the equations of motion for compressible, viscous and heat-conductive Newtonian fluid in a bounded domain, with small initial data and external force, and boundary conditions of zero velocity and constant temperature. We also show that the solution decays exponentially to a unique equilibrium state. The proof uses an energy method similar to the one used in our previous results on the pure initial value problem plus some new techniques for estimates near the boundary.		